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FLAT-SPHERE PERSPECTIVE

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Abstract—*In this paper a new system of visual representation, Flat-Sphere Perspective, is developed. With Flat-Sphere Perspective, an image of the entire visual space around an observer can be represented on a flat surface. The system integrates in a coherent and continuous image the many instantaneous views a human observer perceives as he turns his head.*

This new perspective system furnishes the contemporary artist with a new representational format that corresponds more closely to the geometrical and perceptual principles governing visual perception than any other system hitherto devised.

I. INTRODUCTION

For the last 500 years, human beings have been creating two-dimensional representations of the visual three-dimensional world guided by a system of representation called Central Convergent Perspective (CCP). Developed primarily during the Renaissance, CCP has come to us as a set of concepts and principles which provide: (1) a geometrical analysis of the human visual field, and (2) a way to represent that visual field on a surface. CCP's considerable success in representing the visual world is evident in that many pictures constructed with CCP can actually 'fool the eye' of an observer.

Although the merits and usefulness of CCP as a system of visual representation are undeniable, it has become increasingly clear that the system is both limited and flawed. In this paper a new system of perspective, Flat-Sphere Perspective, will be developed which, it will be argued, captures more adequately than CCP the range and geometrical nature of the visual world.

In the first section of this paper some of the shortcomings of CCP are briefly examined. In the following sections, Spherical Perspective and then Flat-Sphere Perspective are developed.

In a second study, Flat-Sphere Perspective will be refined and developed further into Polar Perspective. With Polar Perspective it is possible to create perspective images of more than three dimensions which are perfectly coherent and nonambiguous.

II. THE SHORTCOMINGS OF CENTRAL CONVERGENT PERSPECTIVE

A very unremarkable, yet fundamental, fact about human visual perception is that a person can turn his gaze in any direction around him—up, down, left, right, front, back—and find the visual world. This simply means that, as human beings endowed with vision, we are completely surrounded by a visual world. It is as if we were in the middle of a balloon, such that in any direction we turn our gaze we find that balloon (Fig. 3, top). This imaginary balloon is usually referred to as the sphere of vision.

But we can see the visual world surrounding us only successively in discrete chunks. We cannot see the totality of the surrounding world at once because our sense of sight is such that it furnishes us at any single moment with only a partial view of the total visual world. Although we see only one portion of the entire visual world at a time, the visual world does not present itself as a succession of disconnected chunks of space, but as one and the same visual space.

If the span of our instantaneous visual field were larger than it is, as one finds it to be in certain animals, then we would see more of the sphere of vision at one time [1]. In principle, there is no reason that we could not imagine ourselves, or another sort of being, as having evolved with a visual apparatus capable of perceiving the totality of the surrounding visual space at once.

How would the world look to us if we could see all around at once? What kind of perspective is created on a spherical perceptual image-surface? Is it possible to create a flat image corresponding to the all-around perception and, if so, how? To answer these questions, another question must first be answered. Is it possible to analyze adequately the perspective of the spherical visual field with CCP?

The answer to this question is no, CCP cannot capture the range and geometry of the all-around perception because CCP developed from an analysis of the limited, instantaneous visual field which CCP conceived as a flat surface. CCP wrongly assumed the geometry of the instantaneous visual field to be that of an Euclidean flat surface, commonly called a picture plane (Fig. 1). CCP's picture plane will never be able to capture the range of view of the sphere of vision, for it should be clear that through the window that a picture plane opens, however large it is conceived to be, an observer will be able to see only one side of

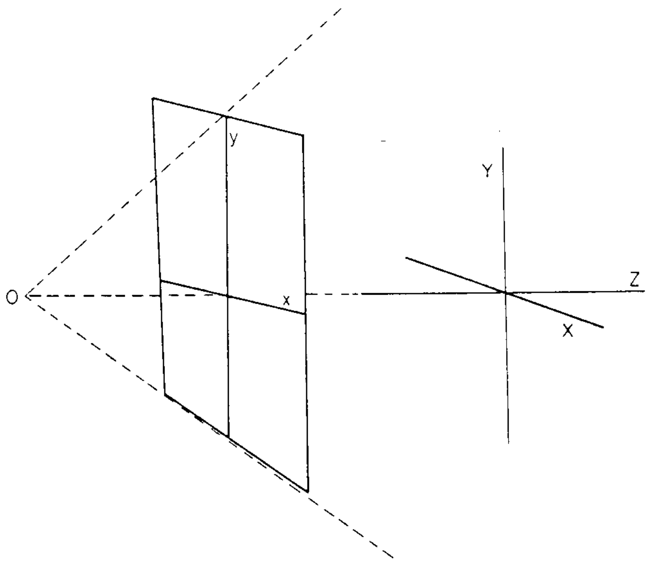

Fig. 1. Central Convergent Perspective's picture plane and the three spatial axes.

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the entire visual world at a time. On the other hand, the spherical visual field allows an observer to see in every direction at once.

Apart from the geometry of the field itself, CCP also analyzes roughly two of the spatial dimensions of the three-dimensional perceptual image that appear on the visual fields as dimensions parallel to the two dimensions of the picture plane. Of the three illusionistic spatial dimensions (*x,y,z*) represented in Fig. 2, top (the *perspective schema* of CCP) [2], the *x* and *y* dimensions are parallel to the two dimensions of the representational plane (or picture plane). The *x* and *y* dimensions of the three-dimensional image are parasitic on the two dimensions of the plane of representation. Quite differently, the dimension of depth (*z*), which does not exist on the representational plane, is given totally by illusionistic means, i.e. the depth dimension is not parallel to any of the dimensions of the image-surface so it appears to vanish into the distance, creating the phenomenon of convergence.

CCP necessarily construes the three-dimensional space present in the visual field as a three-dimensional Euclidean Space because the Euclidean geometry of the picture plane is automatically transferred to the three-dimensional image when two of the dimensions of the image are given as parallel to the Euclidean dimensions of the picture plane. Consequently, with the principles and concepts of CCP it is impossible to create an image of a three-dimensional space which is not Euclidean [3].

CCP gives rise to anomalous images such as the famous Column Paradox. These anomalous constructions exist precisely because CCP conceives the instantaneous visual field as flat. When these constructions are conceived as projected on a concave visual field (part of a sphere with the observer at the center), they do not exhibit the anomalies that are present when they are projected on a plane [4].

CCP's picture plane functions like a window into another world because it creates arbitrary frames that delimit the 3-dimensional space that always extends beyond the limits of the picture plane. (The *x* and *y* lines of the CCP schema can be extended infinitely.) The circumscribed picture-plane, by necessity, will never provide more than a partial view of the Euclidean space of its image. So, although the arbitrarily bounded picture plane does capture the peculiar window-like nature of any instantaneous human visual perception, it is unable, in principle, to contain an image of the whole of a visual space. Since CCP's picture plane cannot possibly stand as a model for an all-around perception of the world, how should the visual field of an observer who sees instantaneously the entire visual space surrounding him be conceived?

The answer is that this visual field should be conceived as a transparent spherical surface, the balloon mentioned earlier, with the observer in the center of the sphere (Fig. 3, top). Unlike

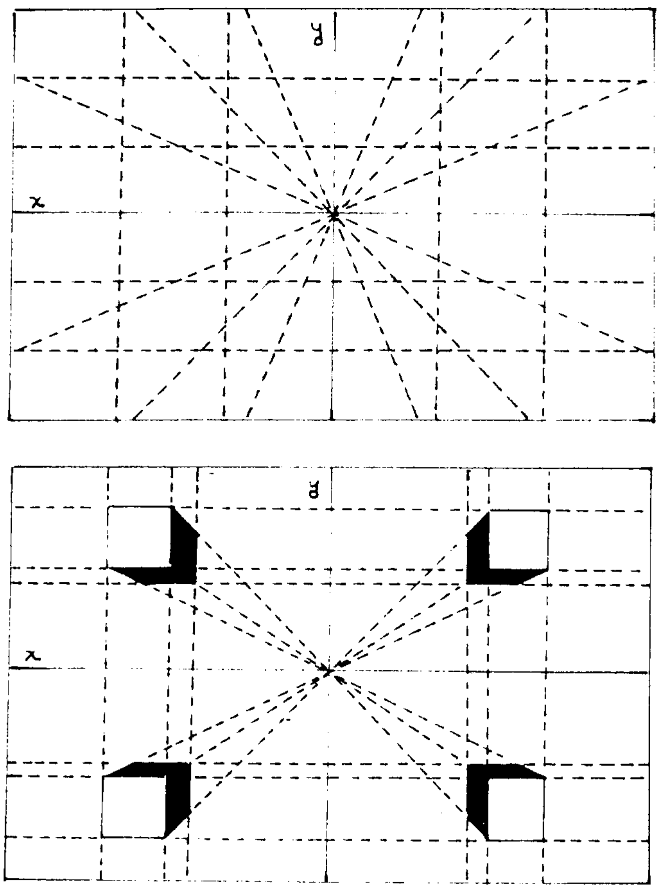

Fig. 2. (Top) Central Convergent Perspective schema. (Bottom) Four cubes in Central Convergent Perspective.

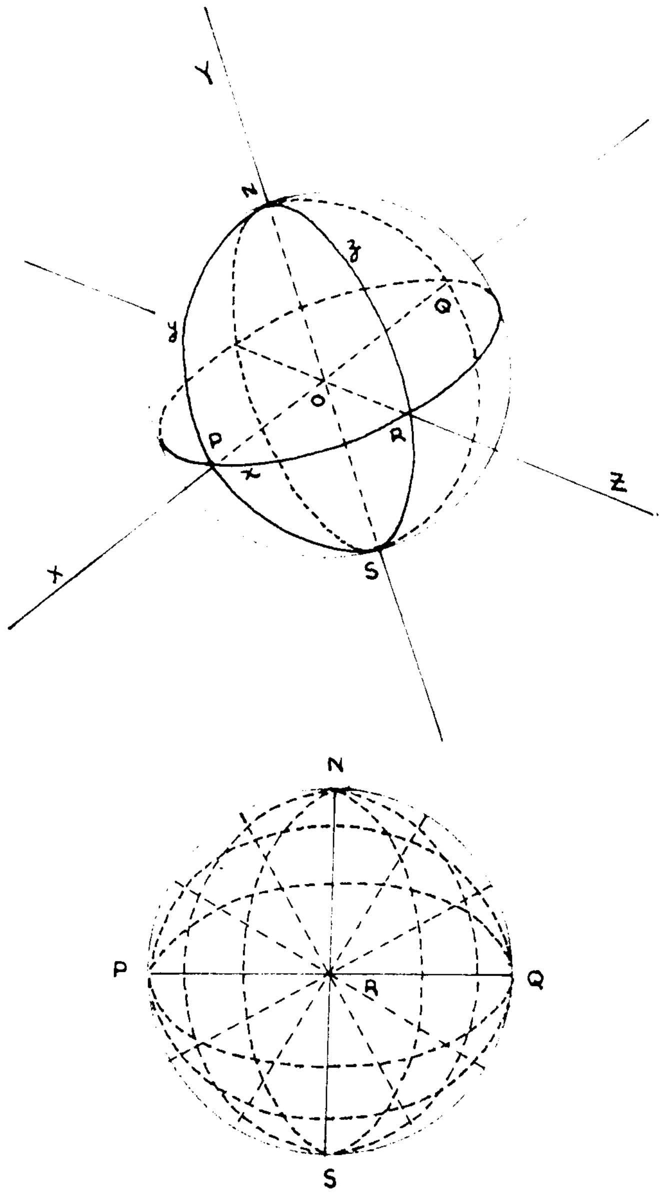

Fig. 3. (Top) The sphere of vision and the three spatial coordinates. (Bottom) The Spherical Perspective schema on an opaque sphere.

the CCP model, this model captures the following essential features of the perception of visual space that we experience:

(1) As an actual observer encounters the visual world in any direction, he turns his gaze; similarly, the observer in the center of the sphere encounters the sphere surrounding him.

(2) As the surrounding visual world exhibits no boundaries or breaks of any kind, similarly, the spherical surface surrounds the viewer with perfect and boundless continuity.

(3) Actual observers, after a full turn of their gaze around visual space, find that their gaze has returned to the same visual spot; similarly, after a full turn, the observers find their gaze returns to the same spot on the spherical surface.

The spherical image-surface represents more adequately than CCP's picture plane the perceptual and spatial relations between a human observer and the visual world around him. The sphere, unlike the plane, captures the fundamental fact that human beings do actually see in every direction although it takes them some time to do it. Every and any instantaneous view should be conceived as a portion of a more fundamental visual apprehension of the sphere of vision as a whole. Consequently, the instantaneous visual field should be conceived, not as a flat plane, but as a concave surface that is a portion of a spherical surface.

III. MATHEMATICAL AND GRAPHICAL MAPS

Although there exists in the literature a clear understanding that the sphere is a more adequate model for the visual field, there is to my knowledge no development of a perspective system for a spherical image-surface or a perspective system by which the spherical surface can be mapped on a plane; namely, Spherical Perspective and Flat-Sphere Perspective. One important reason for this state of affairs is the well established mathematical insight that the sphere and the plane are topologically different surfaces, which means that a spherical surface cannot be mapped onto a plane surface with perfect uniqueness or continuity. Without contradicting the fact that, *mathematically* speaking, a sphere cannot be mapped onto a plane, it shall be seen that from the purely *graphical* point of view it is possible to flatten a sphere. Accordingly, Flat-Sphere Perspective, which is a system of visual representation, shall be developed using concepts of graphical representation. The concepts of point, line and surface are to be conceived as graphical entities and, as such, are to be distinguished from their mathematical counterparts. *Graphical* points and lines are visual or representational entities which have magnitude and obtain an appearance on a representational field. This field, in turn, is conceived as a homogeneously elastic surface. Graphical points and lines are consequently thought to be stretchable in any direction along this elastic film.

It is possible to create on a plane an image or representation of a sphere where there exists a one-to-one correspondence between the graphical points of the sphere and the graphical points of the flat-sphere and where the spherical neighbourhood relations between graphical points are retained on the planar image. Nevertheless, Flat-Sphere Perspective, which is developed in this essay, creates graphical representations where only the uniqueness requirement is completely fulfilled. In Flat-Sphere Perspective neighborhood relations are altered around one point. Perfectly unique and continuous graphical representations of spherical images on a plane are possible when Flat-Sphere Perspective is developed into Polar Perspective.

IV. SPHERICAL PERSPECTIVE

In order to obtain the schema for the perspective of the sphere, the appearance of the three spatial dimensions projected onto the spherical image surface must be determined. When the three spatial coordinates are projected onto the CCP picture plane, and two of the coordinates are made parallel to the plane, the projection exhibits only one fundamental point of convergence [5]. On the other hand, the schema for Spherical Perspective has six and only six fundamental vanishing points. Evidence for this comes from (1) visual experience and (2) from the geometry of the sphere.

(1) Visual experience shows that Spherical Perspective must have six fundamental vanishing points. Consider the following example. Imagine yourself in interstellar space near three very long ladders that run perpendicular to each other, and that correspond to the three spatial dimensions. One ladder runs below your feet, forwards and backwards. Consequently, you see this ladder vanishing into a point in front of you. Turning your gaze towards the back view, you see the ladder vanishing again towards another point opposite the first. Next, imagine the second ladder running horizontally directly in front of you. Moving your gaze to your right you see the horizontal ladder vanishing into a point, and similarly when you turn your gaze to the left you see the ladder vanishing into a point opposite the right one. The case is exactly the same with regard to the vertical ladder. Looking upwards and downwards you see the ladder vanishing at two points poles apart, one above you and the other below you. Thus, visual experience tells us that an all-around view of the surrounding visual space must create an image with six and only six fundamental vanishing points.

(2) On a plane surface only two straight lines can be drawn that intersect each other perpendicularly. The geometry of a spherical surface, on the other hand, allows one to draw on its surface three straight lines (great circles) perpendicular to each other (great circles *x, y, z* of Fig. 3, top). Moreover, these three great circles intersect each other at six points evenly spaced on the surface of the sphere. These are the points where the three spatial axes (X, Y, Z) intersect the sphere (points P, Q, R, T, S, N of Fig. 3, top).

The Spherical Perspective schema is created on the surface of the sphere by drawing the three sets of great circles that correspond to the X, Y, and Z spatial coordinates. Each set of great circles converges at two points diametrically opposite to one another. Figure 3, bottom, exhibits the appearance of the Spherical Perspective schema on an opaque sphere. Notice that the three sets of great circles create six and only six points of convergence exactly in the arrangement that out visual experience requires them to be.

V. FLAT-SPHERE PERSPECTIVE

Since our perception of the entire visual world must be conceived as an image projected on our spherical image surface, the representational artist is faced with the task of transferring this spherical image onto a flat surface. For it is only when the spherical image is flattened that we can see the whole image at a glance. It would be absurd for an artist to create, in this spirit, a representation of a spherical perceptual image on the inside walls of a large spherical room, for example. Since we cannot perceive the sphere of vision at once we could not see the all-round painting of the scene either. So, the artist must deal with the problem of how to flatten the spherical image if he is to create a representation that fits our narrow, instantaneous sense of sight. But how can a sphere be flattened?

A familiar method of mapping a sphere onto a plane is through stereographic projection. This method is nevertheless useless to the artist, because the full sphere maps only on an infinitely large plane. In order to obtain a flat and finite representation of a full spherical surface, I propose mapping the sphere onto a plane through a purely graphical method. The sphere and the plane are made to touch at points *S* and *S′* respectively. Now conceive of the spherical image as a homogeneous elastic film, and imagine flattening it on the plane after piercing it at point *N*, as illustrated in Fig. 4.

By this method the entire spherical image-surface is flattened onto a finite plane—there is no part of the spherical image surface which is not present in the flat-sphere image. (Clearly, the topological alteration that the sphere undergoes when it is pierced at point *N* changes the graphical neighbourhood relationships in the vicinity of point *N*. This and other related issues are discussed in the last section of this paper.)

The flat-sphere obtained by this method is not an arbitrarily bounded representation, like a CCP representation. Far from being a window-like representation, the flat-sphere obtains its boundary at the very point where the entire sphere of vision has been completely represented. When graphical point *N* is pierced and the sphere is flattened, graphical point *N* stretches latitudinally so as to become the whole circumference of the flat-sphere (graphical point *N′*). All graphical points of the sphere will stretch latitudinally except point S [6]. Figure 5 illustrates the flat-sphere schema, i.e. the spherical perspective schema flattened in the manner explained above. Figure 6, which was created on the basis of this Flat-Sphere Perspective schema, exhibits 24 cubes in Flat-Sphere Perspective.

When we look at an image created with Flat-Sphere Perspective, we have the extraordinary experience of seeing—by virtue of the illusion created by the representation—in every direction at once.

When a spherical image is flattened, two things are mapped onto the representational plane: (1) the spherical surface (the graphical set of points which constitute the spherical image surface), and (2) the perspective image on the spherical surface. The geometrical properties of the sphere are automatically mapped onto the plane when the spherical image, or the

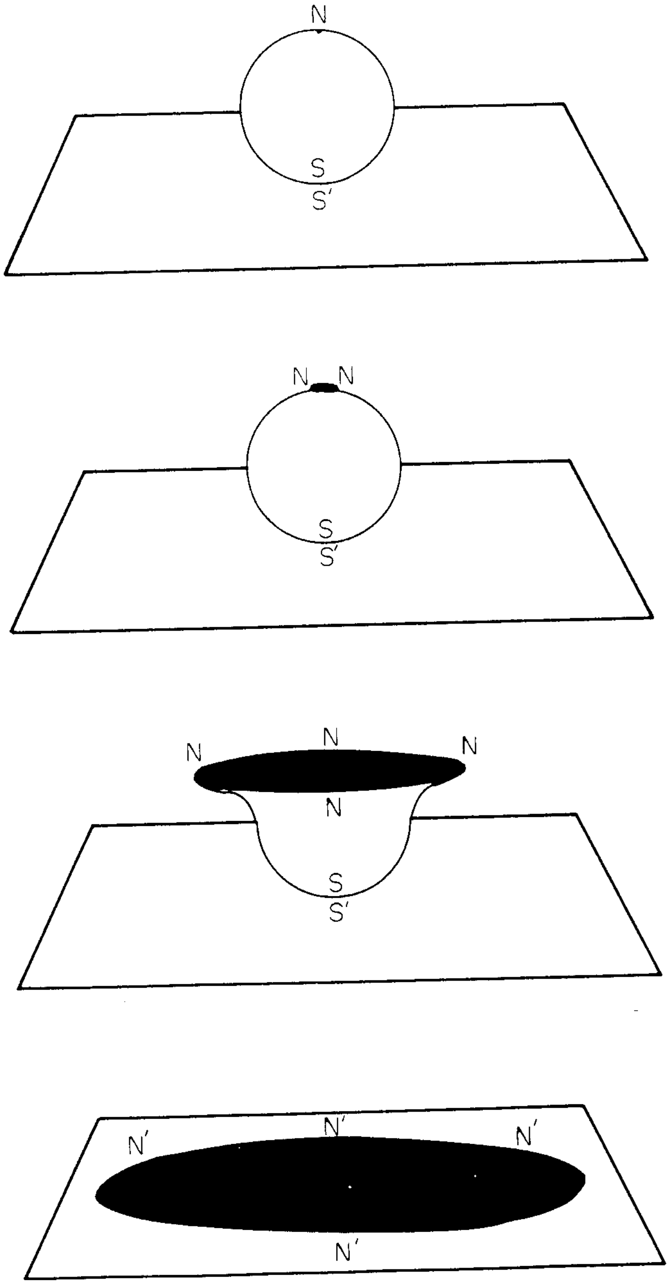

Fig. 4. Graphical representation of a sphere being flattened.

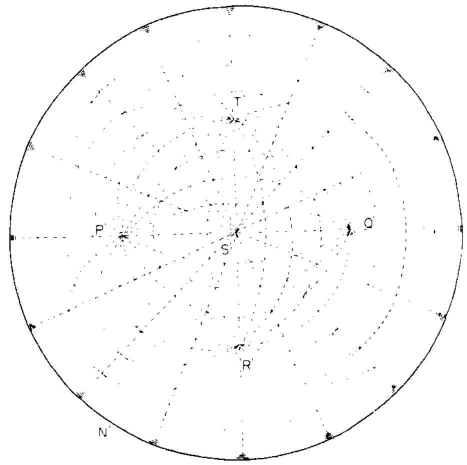

Fig. 5. Symmetrical representation of the Flat-Sphere Perspective schema.

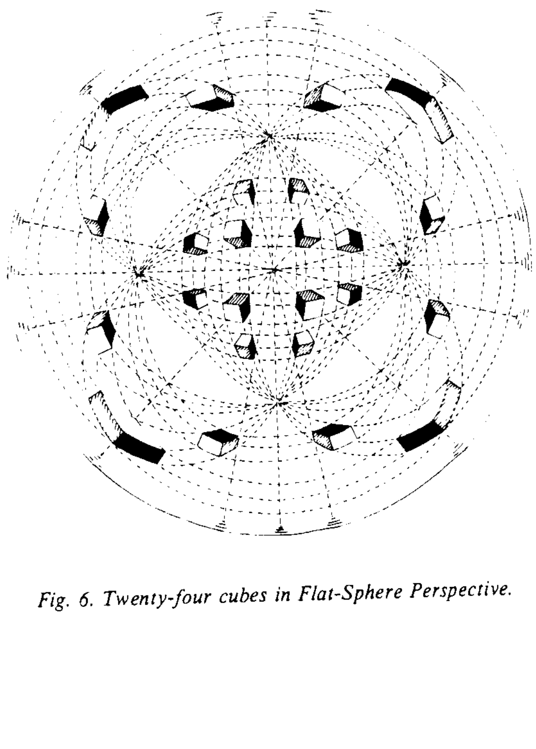

Fig. 6. Twenty-four cubes in Flat-Sphere Perspective.

spherical schema, is mapped onto the representational plane. This happens because the spherical image itself, or the spherical schema itself, carries implicitly the geometry of a spherical surface. Consequently, mapping a spherical perspective image onto a plane implies mapping a spherical surface onto a plane. The schema of Fig. 5 is thus a representation on a flat surface of a spherical surface on which there is a perspective grid.

The examination of the last illustrations should make clear how the three sets of great circles create a three-dimensional, entirely illusionistic space. Unlike CCP, in Spherical Perspective and in Flat-Sphere Perspective none of the three dimensions of the image is parallel to the representational plane or parallel to the spherical image-surface. The three spatial dimensions represented by Spherical Perspective and Flat-Sphere Perspective are dimensions that vanish in the distance, creating the phenomenon of convergence. Spherical Perspective and Flat-Sphere Perspective free the spatial dimensions of the image from the dimensions of the image-surface.

For this reason, the non-Euclidean properties of the spherical image-surface are *not* transferred to the image itself. The geometry of the three-dimensional space represented in a Flat-Sphere Perspective image is therefore not necessarily non-Euclidean. Notwithstanding the curvilinear appearance of the Flat-Sphere Perspective schema, the space represented by those curved lines may be Euclidean or non-Euclidean. The lines of the Flat-Sphere Perspective schema are curved not because they represent a non-Euclidean three-dimensional space, but because they are great circles of the sphere which have been flattened. Flattened great circles become curved lines on a plane. Consequently, the curvilinear appearance of the Flat-Sphere Perspective schema is a manifestation of the non-Euclidean nature of the spherical visual field and not a manifestation of the spatial properties of the physical world represented in the image that appears on the field. In other words, even if physical space were Euclidean, its image projected onto a spherical visual field and then flattened would appear curvilinear in the manner illustrated. To repeat, a Flat-Sphere Perspective image implies nothing about the geometry of the space it represents; this space may be Euclidean or non-Euclidean. In fact, it will be shown that the great representational potential of Flat-Sphere perspective rests precisely on this fact, i.e. on the fact that, unlike CCP, it is a system of representation where the spatial dimensions of the image itself are completely independent from the dimensions of the image-surface.

It is important to realize that a spherical image-surface may be flattened in any direction. In the illustrations, the point from which the sphere was flattened, point *S* (*the central mapping point*) coincides with one of the fundamental converging points of the schema. However, this need not always be the case. One may choose to map a sphere from any point on its surface. Thus, any point of the image on the spherical surface may become the central mapping point. In each case, the resulting flat-sphere image will have a different appearance than in the other maps. Hence, one spherical image can, in principle, yield an infinite number of isomorphic flat-sphere images.

A very conspicuous feature of the Flat-Sphere Perspective schema of Fig. 5 is that five of the fundamental vanishing points appear as points of convergence, but vanishing point *N′* appears as a *divergent* point. Due to the distortion that the sphere undergoes during the mapping procedure, vanishing point *N* becomes divergent point *N′*. This feature of the flat-sphere image is unlike the visual experience that the all-around viewer would have, for he normally sees at point *N* a converging phenomenon similar to the one he sees in the five other vanishing points. Flat-Sphere Perspective, which closely duplicates the appearance of the spherical image in many respects, fails to do so in this respect.

Of the infinite number of mappings possible, only six mappings create the phenomenon of perspective divergence. It occurs only when the central mapping point coincides with one of the fundamental vanishing points. But in those mappings where this coincidence is not present, the Flat-Sphere Perspective schemas obtained are asymmetrical in appearance and exhibit the six fundamental vanishing points as convergent points (Fig. 7).

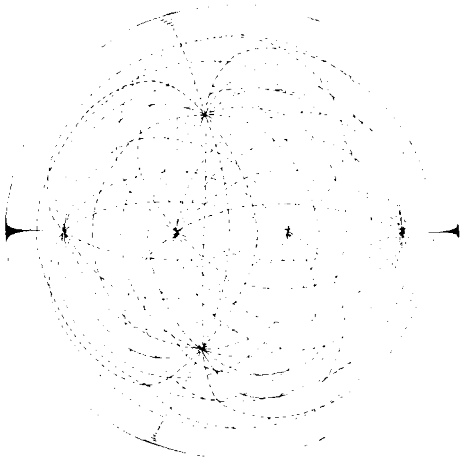

Fig. 7. Asymmetrical representation of the Flat-Sphere Perspective schema.

In these asymetrical mappings the point at the north pole of the sphere does stretch so as to become the periphery of the flat-sphere, but it does not become a point of convergence or divergence since the schema lines do not intersect at this point. The phenomenon of divergence or convergence exists only in reference to the appearance of the schema lines; the phenomenon of stretching, on the other hand, refers to a surface on which these lines may or may not appear.

VI. MAPPING AND DISTORTION

When a Spherical Perspective image is mapped onto the representational plane, it undergoes various transformations. These will be examined under two headings: (1) stretching, and (2) topological transformations. But before discussing these, it is necessary to clarify the relationship between a perspective image and a perspective schema.

A *perspective image* is a two-dimensional appearance which creates in the observer the illusion of depth by virtue of the general linear organization of its elements. Examples of perspective images are conventional photographic snapshots, pictures done with CCP (or any other perspective system), and the perceptual image in the visual field of an observer.

A *perspective schema*, on the other hand, does not create the illusion of depth. Rather, it is simply a *two-dimensional graph* which either results from *an abstraction* of the linear perspective of an image, or is created in advance of an image in order to produce it. A perspective schema is a *diagram* that shows three sets of lines on a graphical plane that correspond in visual space to the three spatial dimensions.

A perspective schema provides a *two-dimensional skeleton of the geometrical relations* within a three-dimensional illusionistic image. For example, within a CCP image, objects *vanish* in the distance. This vanishing phenomenon of the three-dimensional visual space is abstracted in the schema as the two-dimensional phenomenon of *convergence*. In other words, vanishing is a three-dimensional visual phenomenon; convergence is a two-dimensional description.

A perspective schema represents the geometrical structure of potentially many images. A perspective schema is a tool which indicates how any number of perspective images can be created. Conversely, an image done in perspective has one and only one perspective schema.

Since a schema is a graphical entity abstracted from but independent of any perspective image, the graphical lines of a schema can be drawn as thinly or as thickly as is necessary to communicate the required information. Although the thinner a graphical line is, while still being visible, the finer a tool for analysis and construction it is, this will not be the only criterion in determining the width of schema lines.

A. Stretching

As mentioned before, when a spherical image is mapped onto the representational plane, all points of the spherical image (except point S) undergo latitudinal stretching, which increases as the distance from point S increases. This stretching affects the *appearance* of an image in two respects: (a) if, in actuality, a physical spherical image on an elastic film is flattened, the image will get blurry toward the edge, and (b) the proportions of the objects in the flattened image will be different from the proportions that these objects have on the spherical image.

1. Blurring of the image. The mapping of the spherical image onto the representational plane has been repeatedly illustrated by the analogy of an elastic sphere that is actually flattened on the plane. This analogy helps in conceptually understanding that the flat-sphere image retains all the visual information during the mapping process. When such a spherical image, which is assumed to be in focus, is actually flattened, it will look increasingly blurry from point S' to point N'. That is to say, outlines and shapes will get less defined and increasingly fuzzy, colors and values will become washed-out, and textures will flatten and homogenize as they approach point N'.

About this physically flattened blurry image, the following considerations need to be mentioned. In the first place, this is not the only kind of flat-sphere image that can be obtained, and in fact, it is not the flat-sphere image primarily to be considered here. The flat-sphere image to be considered is an image that is as sharp and fresh as the original image. This flat-sphere image can be re-created with the aid of the proper flat-sphere schema.

This is possible by first abstracting the Spherical Perspective schema from the original spherical image. Then by mapping this schema onto a plane, after choosing a desired mapping direction, the corresponding Flat-Sphere Perspective schema is obtained. Based on this schema, an artist can 'flesh in' or reconstruct on a plane the whole spherical image with as much sharpness and freshness as desired. The flat-sphere image he obtains via the schema is as much a mapping of the original spherical image as the physically flattened blurry image is.

It must be stressed that the actually flattened blurry image is an image that retains the geometrical structure present in its spherical state. As the perspective structure of a photographic image is not altered by printing the photograph in or out of focus, similarly the perspective of the spherical image is not altered by its becoming blurry as a result of the physical flattening. This flattening affects its appearance, not its perspective structure. That is to say, from the geometrical analysis of the spherical image and from the geometrical analysis of the blurry flat-sphere image, the same perspective schema is abstracted—the only difference being that the two schemas are on topologically different surfaces.

The blurring phenomenon affects only the flattening of actual physical spheres; it does not affect the flat-sphere images created via perspective schemas. The artist therefore is not constrained by the mechanics of some actually flattened hypothetically elastic sphere.

2. Distortion. The latitudinal stretching, however, does affect the *appearance* of the schema and consequently the appearance of the image created on its basis in other ways than the blurring of the image. But again, this happens without altering the perspective structure of the image.

A spherical image undergoes latitudinal stretching when mapped onto a plane, regardless of whether the image is physically flattened or mapped onto the representational plane via a schema. This stretching is manifested in the different proportions that objects in the flat-sphere image obtain. Objects elongate latitudinally, and this elongation clearly changes the *appearance* of these objects and also changes the *appearance* of the flat-sphere schema in comparison with the spherical schema. For instance, 'straight' lines in the spherical schema (the great circles) appear as curved lines in the flat-sphere schema. Taking the spherical image as the standard of appearance, the stretched flat-sphere image is obviously a distorted image.

But this distortion should not be viewed in a negative light, as the price paid for a flat representation of a spherical image. On the contrary, it is precisely in virtue of this distortion that it is possible to transfer the spherical image onto the flat surface without changing its internal perspective structure. In general, it is not possible to represent a spherical surface on a flat surface without distortion. If an image on a spherical surface is to retain its internal geometrical structure when mapped onto a plane, the image must undergo certain transformations (distortions) which together exhibit the original spherical surface on which the image appeared. To say that visual space surrounds us is to say, among other things, that its visual representation on a plane must be a curved, distorted image. Consequently, these distortions are the *visual manifestations of the spherical nature of visual space* as it is displayed on a plane, and, as such, these distortions should be welcome as a yet unfamiliar prize [7].

The *structure* of the flat-sphere schema must reflect the stretching distortions that the objects represented in the image undergo. But the lines of the schema *themselves* are *in general* unaffected by the stretching. As was mentioned earlier, the lines of a schema can be drawn with whatever width the analysis or construction at hand requires. However, there is one sort of schema line that in particular is affected by the stretching of the image.

When a sphere is mapped onto a plane, the schema of this flat-sphere image has, by necessity, one graphical point of a different magnitude from all other points of the schema. This is point N', the circumference of the flat-sphere schema. Point N' is a graphical point larger than any other point of the schema. Since the geometry of the schema calls for the drawing of segments that run from point S' to point N', then such segments will have to vary their width in order to go from small point S' as a whole, to large point N' as a whole.

If the flat-sphere schema were constructed with consistently thin lines, as illustrated in Fig. 8, top, then the appearance of the S'–N' lines would be that of lines going from point S' to a point of point N', i.e. from S' to N'_1 or to N'_2 or to N'_3 or to N'_4. Graphically, this representation does not duplicate the geometry of the sphere, for it does not make it visually evident that N' is but a point. Moreover, this representation maps great circles S–N–S as (Euclidean) straight and bounded segments. Of course, we can *understand* that these segments represent closed and boundless lines (i.e. great circles) because they end at the circumference of the schema which we *understand* to be just a point. That is to say, we understand that from point N'_1, one is logically entitled to go to point N'_3, which is on the other side of the representation. But we cannot *see* this in the appearance of Fig. 8, top. All other lines of the flat-sphere schema, which are also mappings of great circles, appear as curved lines that return to themselves. So, it is desirable that the N'–S'–N' lines should also appear as closed lines in the flat-sphere schema. To the artist, who is interested not only in understanding an image, but in the visual appearance of it, this disparity is of importance. The question then, is how could lines N'–S'–N' be drawn so that

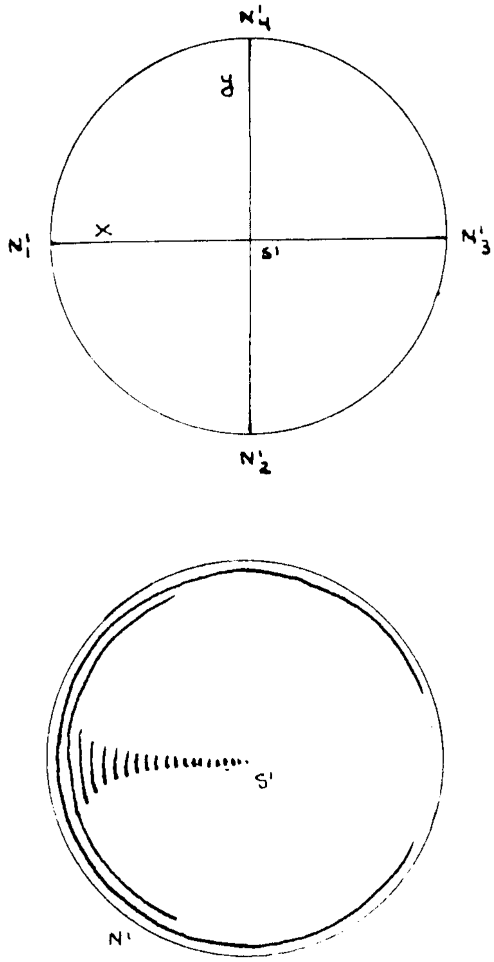

Fig. 8. Two inappropriate representations of schema lines. (Top) The x and y axes appearing without stretching. (Bottom) Simplified representation of the actual stretching that a S–N line on a sphere undergoes when physically flattened on a plane.

they conform (1) with the geometrical requirement of going from the whole of point S' to the whole of point N', (2) with the general curvilinear and closed-path appearance of the rest of the schema lines, and (3) with the basic requirement of any schema line of being thin enough for analysis and construction of images.

To answer this question, consider first what would be the appearance of an actual pencil line drawn on the sphere fron S to N, after the sphere has been physically flattened. If this line is thought of as made of a succession of contiguous graphical points, then point S, when the sphere is flattened, becomes point S' and undergoes no stretching [8]. But the point next to point S in the S–N line will undergo a very small amount of latitudinal stretching. The point next to this one will undergo a slightly greater stretching until the last point, point N, undergoes the most stretching, becoming the circumference of the representation. This incremental stretching of points is illustrated for segment S'–N' in Fig. 8, bottom, in a very simplified manner.

Should flat-sphere schemas be drawn with the S'–N' segments stretched as illustrated in Fig. 8, bottom? Of course they should not be. Such lines are far too thick to be helpful as schema lines. After all, with a schema one should be able to make graphical distinctions as fine as possible. Moreover, one should be able to analyze with the schema lines the very phenomenon of stretching as it affects image lines. With an expanded line as that of Fig. 8, bottom, only very crude drawings could be done, and it would be impossible to analyze the stretching phenomenon itself.

Yet there is no reason for concern. The area of ambiguity created by the stretched S'–N' line of Fig. 8, bottom, can be reduced so that the line meets the geometrical requirement of going from the whole of point S' to the whole of point N', and it meets the practical requirement of being a very thin line if necessary. The schema of Fig. 5 is constructed with such lines. They are normal thin lines during most of their paths, but as they approach N', they stretch to cover all of N' as required. One is justified in drawing such lines because, as noted before, they are graphical lines of a schema and not mathematical lines or actual pencil lines of a sphere that has been physically flattened.

In this schema, lines N'–S'–N' actually *appear* as lines having a closed and somewhat curvilinear path as all the other lines of the schema do. This gives the schema a more coherent and harmonious appearance, for the schema now displays *visually* that circumference N' is just a point in the image.

B. Topological transformations

The sphere and the plane are topologically different surfaces. The sphere is a closed and finite surface while the plane is an open and infinite surface. Therefore, it is impossible to map a sphere onto a plane with perfect continuity and uniqueness. As Hans Reichenbach points out: 'a unique and continuous mapping is possible only for geometrical structures having the same topology' [9].

1. Continuity. A sphere cannot be mathematically mapped onto a plane with perfect continuity because neighborhood relations are disturbed around one, and only one, point. This is mathematical point n at the north pole of the sphere. Clearly, the notion of piercing an actual sphere, which is necessary before it gets flattened, implies some tearing of the spherical surface. In topology, the piercing is understood as the removal of a mathematical point (point n) which makes the 'spherical' image-surface no longer a closed surface, but an open surface. Around point n, neighbourhood relations are altered, i.e. points around mathematical point n are separated. The altered neighborhood relations around this point are conveyed into the graphical representation of the flat-sphere. To elaborate, imagine that graphical points A and B on the sphere are adjacent to, but at opposite sides of graphical point N, so that A and B are in immediate proximity on the sphere, with only point N between the two.

On the flattened sphere, however, Points A' and B' will appear far apart from each other at opposite ends of the representational plane. No matter how stretched points A' and B' may appear on the flat-sphere, they still appear, generally speaking, at opposite sides of the representation. Clearly then, around point N' the graphical as well as the mathematical neighborhood relations have been disturbed. Nevertheless, it shall be shown in an ensuing paper that these graphical neighborhood relations can be restored by the creation of a polar image.

It is important to understand that the elimination of mathematical point n does not cause any visual information to be left out of the mapping. Since the mathematical point is extentionless, its removal takes nothing away from the spherical visual expanse. Point N' in the above discussion is not meant to represent the mathematical point that is eliminated. Rather in these discussions, N' must be conceived as the mapping of graphical point N on the sphere, which is actually a small dot on the north pole of the spherical surface. The eliminated mathematical point must be conceived accordingly as the center of graphical point N. In terms of the piercing analogy, this means that the sphere is pierced in the center of graphical point N.

2. Uniqueness. A perfect mathematical mapping requires that there should be a one-to-one correspondence between the set of mathematical points on the sphere and the set of points on the plane. Since a sphere cannot be flattened without first removing a mathematical point from it, it is clear that the uniqueness requirement cannot be fulfilled in the attempt to create a flat mathematical map of a sphere. Removed point *n* of the sphere will have no corresponding point in the plane.

But on the other hand, a graphical mapping of a sphere onto a plane is possible with perfect uniqueness, i.e. for any graphical point on the sphere there is a corresponding graphical point on the plane.

It may be thought that because graphical point *N'* is larger than graphical point *N*, there is not a one-to-one correspondence between these two graphical points. That is to say, because it is possible to identify many other points within point *N'*, then there actually are many points comprising *N'* on the flat-sphere corresponding to graphical point *N* on the sphere. But this is not really the case. The fact that within enlarged point *N'* many other graphical points could be identified does not affect

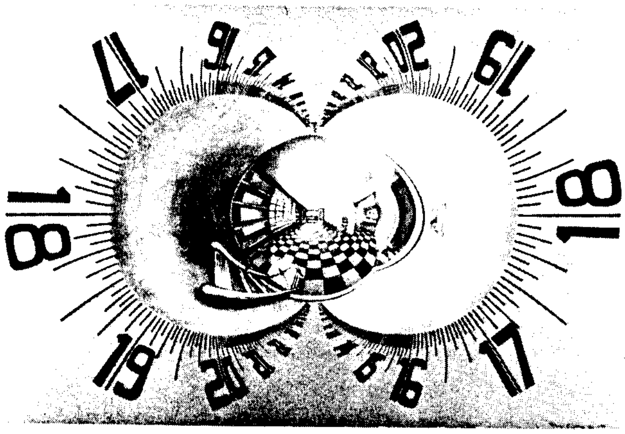

Fig. 9. 'The Measure of All Things', five-color lithograph, 60 × 90 cm, 1979.

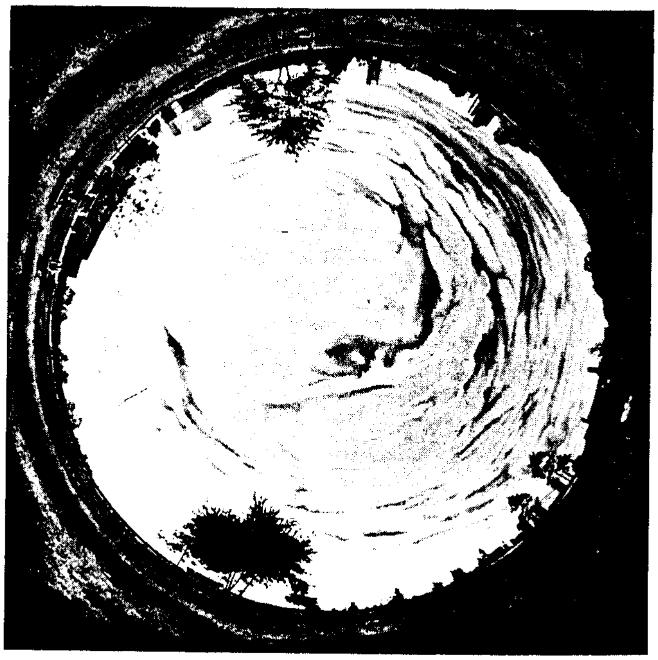

Fig. 10. 'The Sky, Noon', oil on canvas, 1.7 × 1.7 m, 1980.

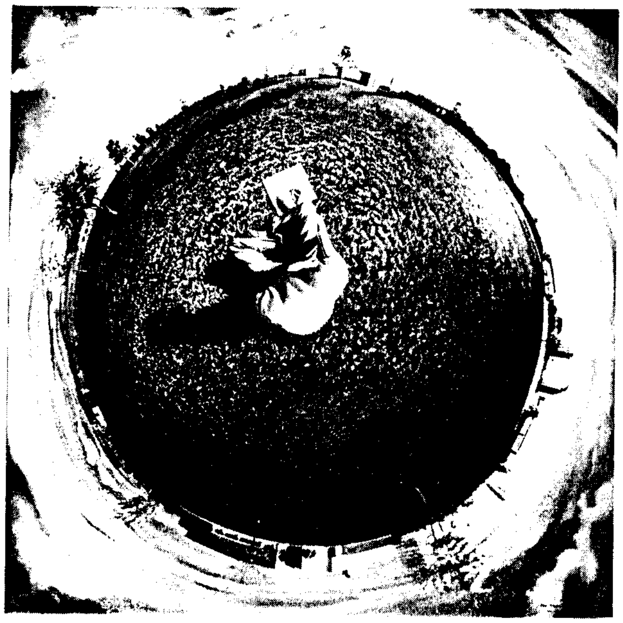

Fig. 11. 'The Planet, Early Morning', oil on canvas, 1.7 × 1.7 m, 1980.

the uniqueness with which point *N* is mapped. After all, within the small graphical point *N* on the sphere, many other points could be identified as well if the point is examined with a magnifying glass. Within *any* graphical point, it is possible to identify—at least in principle—other graphical points.

To reiterate, having to remove a mathematical point from a sphere before it is possible to map it onto a plane implies that the 'flat-sphere' obtained is not a complete mathematical map of the sphere. On the other hand, having to pierce a spherical image before it is flattened does not imply that the flat-sphere image is incomplete with respect to the spherical image. The flat-sphere is a graphical representation of the entire spherical image where the uniqueness requirement is satisfied. Neighborhood relations on this graphical representation, nonetheless, are altered around one point. But this alteration can be restored when a spherical image is flattened in accordance with the principles of Polar Perspective to be explained in an ensuing paper. Four examples of my artwork in which Flat-Sphere Perspective has been used are shown in Figs. 9–11 and Color Plate 2.

REFERENCES AND NOTES

1. The concept of visual field in this work is understood as a two-dimensional expanse on which the perceptual image appears.
2. A *Perspective schema* is a diagram that shows the appearance that three sets of straight, continuous lines have for an observer from his given point of view. Each set of lines is represented as running along one of the three spatial dimensions, i.e. the sets of lines that are perpendicular to each other. In other words, a schema is a two-dimensional grid that defines the appearance of the three spatial dimensions for a given observer.
3. The name Central Convergent Perspective in this work does not refer to the general principles of projective geometry, which can determine the geometry of any image projected on a plane. Rather, the name refers to (1) the inadequate analysis, prevalent since the Renaissance, of a perceptual image and (2) to the principles for the construction of visual representation derived from this analysis, which together comprise Central Convergent Perspective.
4. See for instance R. Vero's *Understanding Perspective* (New York: Van Nostrand Reinhold, 1980).
5. However complex a CCP construction is (two-point or three-point perspective), the illusionistic CCP space has only one *fundamental* vanishing point, i.e. the point where all the lines parallel to the *z*-axis vanish.

6. The sphere need not be considered to stretch only latitudinally. An artist may introduce longitudinal stretching so as to control the appearance that objects will obtain on the flat-sphere image.
7. Throughout this work, the term *visual space* has been used to mark this purely perceptual sense of space apart from *physical space*. Absolutely no claims are being made in this work about the geometrical structure of physical space.
8. This approach to the question was suggested to me by Bruce Leutwyler, to whom I am also indebted for his aid in developing some of these ideas and for editing the manuscript.
9. H. Reichenbach, *The Philosophy of Space and Time* (New York: Dover, 1957) p. 66.

[**Editor's note**—Kenneth R. Adams has described his method of depiction for a 360°-view from a point in space in 'Tetraconic Perspective for a Complete Sphere of Vision', *Leonardo* **9**, 289–291 (1976).]