JOURNAL OF THE INTERNATIONAL SOCIETY

FOR THE ARTS SCIENCES AND TECHNOLOGY

Volume 17 Number 3 1984

Editorial Office: San Francisco State University

Oxford New York Toronto Sydney Paris Frankfurt

Polar Perspective: A Graphical System for Creating Two-Dimensional Images Representing a World of Four Dimensions

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Abstract—The author introduces a system of perspective called Polar Perspective. He explains in nontechnical terms the structure of polar perspective images and how to construct them. Using polar perspective, the artist can create perspective images that represent not only the three spatial dimensions, but also the dimension of time. Moreover, the artist can apply polar perspective to create perspective images that represent in a visually coherent and unambiguous fashion a world of four spatial dimensions.

I. INTRODUCTION

How do the three spatial dimensions of the visual world project (or map) on a surface (or picture)? Imagine a structure of three wooden poles that intersect each other perpendicularly. Each pole represents one of the three spatial dimensions of the world. The person interested in perspective wants to find out what kind of image these three poles create on the visual field of a human observer.

Classical perspective (also called central convergence perspective), which was developed mainly during the Renaissance, gives one explanation. According to classical perspective, the visual image that an observer has in his visual field at a given moment is identical to the image that would be created on a flat window placed between the observer and the object observed. This setup, illustrated in Fig. 1a, is classical perspective's model of visual perception. This model likens the visual field of the observer to a flat surface called the picture plane. For the last 400 years, classical perspective has allowed the artist to create remarkably 'realistic' images of the world that, when placed in appropriate circumstances, were able to fool the eye. Examining Fig. 1a, we can see that the three spatial dimensions of the visual world (axes X, Y, Z) project onto the picture plane a perspective grid with one and only one vanishing point. This is point V, where the projected lines of axes X, Y and Z intersect.

In spite of its remarkable realism, classical perspective creates anomalous images. When we strictly follow the rules of image construction according to classical perspective, we end up creating images that do not accord with the way we actually see the world. This disparity is more evident in some images than others. The notorious column paradox is one example [1]. Such anomalies can be avoided by altering the model of visual perception offered by classical perspective. This can be accomplished by conceiving of the human visual field not as a flat surface, but as a concave surface [2].

We are completely surrounded by the visual world. We can turn our gaze in any direction and see a different portion of the visual world. This is illustrated in Fig. 1b as a spherical surface with an observer at its center. The spherical surface, which replaces the flat picture plane model of the visual field, carries on its surface the image of the entire surrounding visual world. Regardless how narrow our instantaneous visual field, our sphere of vision includes all the visual data of our surroundings. This raises two questions. First, what kind of image do the three dimensions of the visual world project onto this spherical visual field? Second, imagine that we could see all around ourselves at once. How might we represent on a flat surface this visual experience? I have answered the first question with spherical perspective, and the second with flat-sphere perspective [3].

Figure 1b illustrates an observer surrounded by his spherical visual field. The three spatial dimensions are represented by axes *X*, *Y* and *Z*. When these axes are mapped onto the spherical visual field of the observer, they create a perspective grid, the group of lines that organize on the spherical surface the appearance of the three spatial dimensions presented to the observer. This grid has six fundamental points of convergence. Spherical perspective has two advantages over classical perspective. First, spherical perspective dissolves the anomalies that classical perspective gives rise to. Second, spherical perspective organizes in a single continuous image the whole surrounding visual world, rather than only a portion of it.

An artist interested in using spherical perspective might find one important

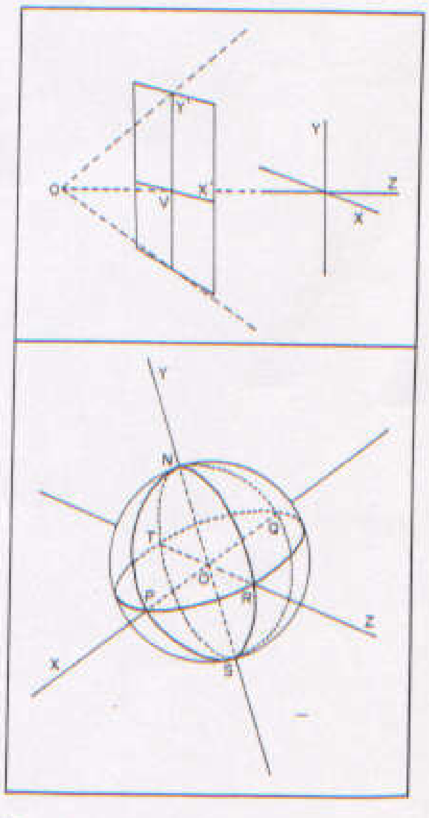

Figure 1 (a) Classical perspective's model of visual perception. The three axes of the visual world—X, Y, Z—map on the picture plane in front of observer O, creating a perspective grid with only one vanishing point, V. (b) Spherical perspective's model of visual perception. The observer O is at the center of his spherical visual field. The three axes of the visual world create a grid with six vanishing points N, S, P, Q, R, T.

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(Received 21 January 1983)

Pergamon Press Ltd.
Printed in Great Britain.
0024-094X/84 $3.00+0.00

shortcoming in the system; spherical perspective images can be created only on spherical surfaces. Consequently, just as we cannot see in one glance the entire visual space that surrounds us, we cannot see in a glance the entire spherical perspective image, whether the image is on the outside surface of a sphere or on the inside surface of a large spherical room. For instance, when faced with a spherical mirror or with a spherical perspective image painted on the surface of a balloon, we can see only one side of the balloon or the mirror at a time. We need to move around the balloon in order to see the rest of the image and around the spherical mirror to see the visual space reflected on the other side of the mirror [5].

Flattening the spherical image results in a perspective image of the entire visual world that can be seen at one glance. This concept led to the flat-sphere perspective system of representing the surrounding visual world on a flat surface. I conceived the sphere of vision to be elastic like a balloon. I could pierce it at a point on its surface and then stretch it into a flat disk. The point at which the sphere is pierced becomes the perimeter of the disk. The disk contains the whole of the spherical image, and it can be seen at a glance.

The spherical perspective image undergoes various transformations during flattening. For instance, the straight lines of the spherical image become curved in the flattened image. Yet 'distortions' like this are actually the visual manifestation on a flat surface of the spherical nature of the visual image. The perspective of the spherical image transferred into the flat-sphere image—the geometrical organization of its perspective grid—remains unaltered. There is, however, one graphical point in the spherical image, and one point only, where its perspective organization is altered by the flattening procedure. This is the point where the spherical image is pierced prior to being flattened. Efforts to overcome this limitation of flat-sphere perspective (which will be explained in more detail later) led me to polar perspective.

Polar perspective is a further development in the field of perspective representation. Polar perspective does not replace flat-sphere perspective. Rather, both flat-sphere perspective and classical perspective are special cases within the more general system of polar perspective. Using polar perspective, the artist can create images that represent not only the three spatial dimensions but also the dimension of time. The system also allows the artist to construct images that represent in a coherent and unambiguous manner four spatial dimensions.

Polar perspective is developed here as a purely graphical system, not as a mathematical system. The concepts of point, line and surface are understood to stand for graphical elements that we can see. A graphical point, far from being a zero-dimensional entity, is roughly a dot on a surface. A line is the sort of elongated trace that an instrument such as a pencil leaves on a surface. In accordance with the elastic surface mentioned above, the points and lines referred to here are graphical entities that can stretch in any direction along the surface in which they appear.

The following sections explain in simple terms the perspective structure of polar images and how to build them. Questions about how to translate this graphical system into a mathematical system and its relationship to theories in physics regarding the fourth dimension are not considered here.

II. CONCENTRIC POLAR IMAGES

An image created with polar perspective is produced when two or more flat-sphere images are connected to form a new, perfectly unified, coherent and continuous image. Figure 2 shows a painting created with polar perspective. Notice that there is a full flat-sphere image in the central portion of this image.

Figure 2. *The Polar Eye*, four-color lithograph, 36 x 24 inches, 1980. This is an example of a simple polar image that contains two flat spheres. Only the enclosed flat sphere appears in its entirety.

This flat sphere is 'surrounded' by another flat-sphere image. (The outer periphery of the surrounding flat-sphere image has been left out for aesthetic reasons. In principle, it could have been represented.) This section will describe how to create a polar image like that of Fig. 2 and the logic behind it.

If a person's visual field were such that he could see all around himself at once, his visual field would exactly correspond to his sphere of vision. For this analysis, we will assume a hypothetical observer whose visual field exactly corresponds to his sphere of vision. Since objects in his sphere of vision may be in motion, our hypothetical observer may have a different image in his visual field at any given moment. Let us imagine this new spherical image placed next to the first image. We can continue adding to our collection of spherical images by making each new sphere represent an instantaneous image obtained on the sphere of vision of our hypothetical observer. The images may be different, but all of them have the same perspective structure.

Figure 3 illustrates a sequence of four such spheres. On the surface of each sphere, we have drawn their perspective structures, so that each sphere displays the same grid of spherical perspective. Notice that each sphere has the same six vanishing points—N, S, T, P, Q and R. Now notice a most important feature of this image: the spheres are not simply one next to another; they are connected in such manner that two contiguous spheres share the same graphical point. For instance, spheres 1 and 2 share point S; spheres 2 and 3 share point N; and spheres 3 and 4 share point S again, etc.

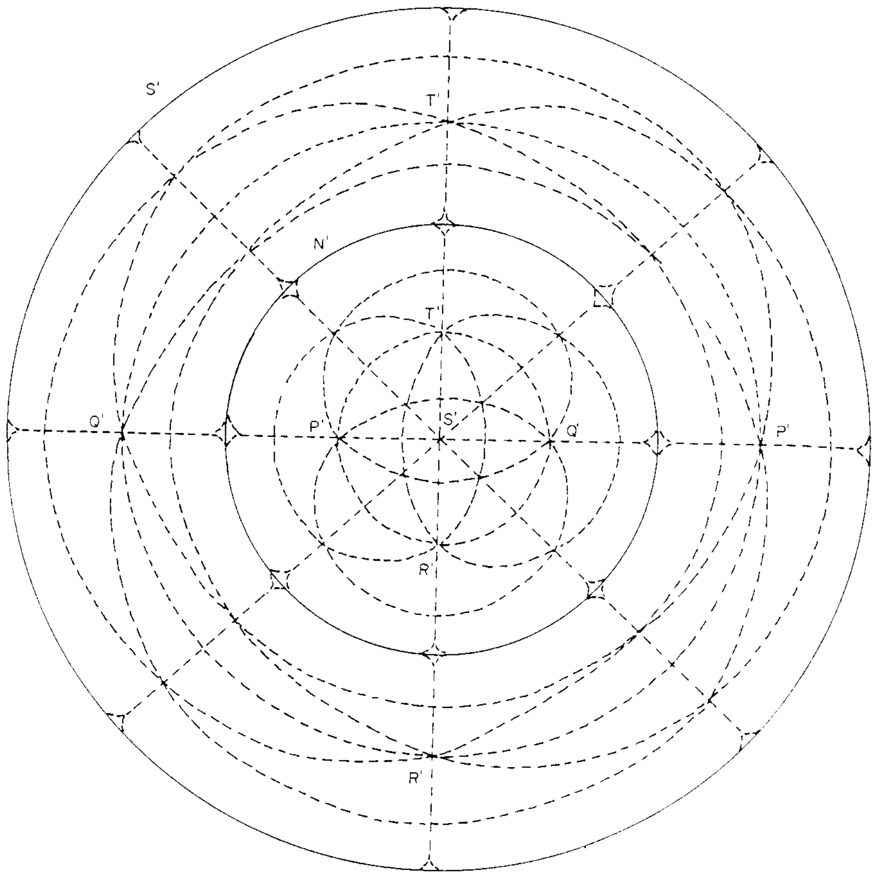

Figure 4. Polar perspective grid of a polar image with two concentric flat spheres.

This string of spheres can be flattened onto a flat surface in a manner similar to the way a single spherical image is flattened in flat-sphere perspective. This is also illustrated in Fig. 3, where we can see sphere number 1 already pierced and in the process of expanding over the representational plane. After sphere 1 has been flattened, point S of spheres 1 and 2 is pierced and sphere 2 is flattened onto the plane, displacing outwardly the already flattened sphere number 1. Next, point N of spheres 2 and 3 is pierced and sphere number 3 is flattened. And so on. In this manner, we obtain on the representational plane a polar image that looks like a sequence of concentric rings. Figure 4 shows the perspective grid of polar perspective obtained in this manner. This figure contains only two flat spheres, but it is possible to continue the sequence by adding as many flat spheres to the grid as we wish.

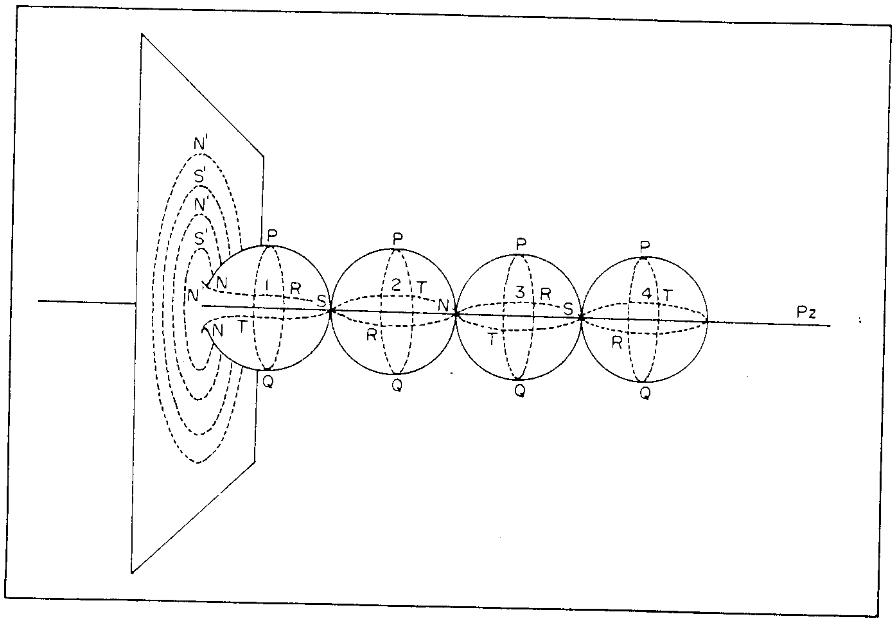

Figure 3. A string of spherical visual fields displaying their identical perspective grids. The spheres are connected continuously because each sphere shares a graphical point with both its neighbors. Sphere 1 has already been pierced and is being flattened onto the representational plane.

The following features are basic to understanding the visual organization of a polar image.

– 1. A polar image is a single and continuous image. It appears to be a ring and a disk: a 'surrounding', ring-like, flat-sphere image with a second 'enclosed' disk-like image in its center. Actually, a polar image is one coherent whole that represents a single sphere of vision; it has no *visual* discontinuities.

Notice that the enclosed flat-sphere image is, in relation to the surrounding one, nothing but its middle graphical point. The surrounding flat sphere in Fig.

2 (or in the grid of Fig. 4), is not a flat sphere drawn onto a ring-like surface that has the enclosed flat sphere as some foreign material filling the hole inside the ring. The surrounding flat sphere is a disk, not a ring; it only happens to have its central vanishing point, N', enlarged as a result of being stretched in the mapping (or flattening) procedure. Within the enlarged, central vanishing point of the surrounding flat sphere, the enclosed flat sphere appears. This enclosed flat sphere is, in relation to the surrounding one, only its central vanishing point. What we have, then, is a representation of a single sphere of vision with some of its points more or less stretched.

Let us use mirrors as an analogy to explain further the relation between two or more flat-sphere images which are part of a polar image. In a polar image the 'enclosed' flat sphere) occupies no visual the 'surrounding' flat sphere, and vice versa. However, this mirroring relationship is such that the mirror itself does not exist as part of the world it reflects.

A mirror ball in our physical world can reflect the entire visual world that surrounds it, but it cannot capture the world inside its own volume. In a polar image, however, the mirror itself (an 'enclosed' flat sphere) occupies no visual space within the world it mirrors, i.e. within the 'surrounding' flat sphere image. Any flat sphere which is part of a polar image is like a spherical mirror of zero dimensions. It is a spherical mirror that has no visual or physical presence in the world it reflects, for this spherical mirror does not hide from view any portion of the visual world it reflects—it does not belong to the world it mirrors.

The single polar image is also a visually continuous image. Point N' is the point of connection between the two flat spheres; both flat spheres share this point. Consequently, the eye can travel from one flat sphere into the other without interruption. The moment the eye arrives at point N' of one flat sphere, it also arrives at point N' of the contiguous flat sphere. In sum, point N' visually bridges both flat spheres into a single and continuous image.

2. The second important feature of polar images is represented in Fig. 4. Notice that the lines which go from point N' to S' to N' again, to S' again, etc. do not look like straight lines because their widths vary in a pronounced way at certain places in the representation. The lines stretch their widths to encompass the whole of stretched graphical points N' and S' as the lines meet these points at the poles of each flat sphere. Any line that belongs to the perspective grid of the flat sphere images and that crosses the boundary between two flat spheres must necessarily stretch circularly in the manner illustrated. It is precisely this stretching of the grid lines that makes the circumference of any enclosed flat sphere function as a vanishing point relative to both the 'surrounding' and the 'enclosed' flat spheres. This feature of polar perspective is particularly relevant to polar images of four dimensions.

3. The concentric polar image can represent not only the three spatial dimensions of our visual world, but also the dimension of time. Given that each flat sphere within a polar image is the mapping of a distinct spherical image occurring sequentially in time, each flat sphere represents a different moment in this time sequence. Consequently, the polar image as a whole is a single, coherent representation of our visual world along a time span.

Like the rings of old trees that record and exhibit the passage of time, a concentric polar image can show—in a discontinuous fashion—the movement of perceived objects in space and time. This is accomplished by making each flat sphere portray the object in a different location as the object changes its position in time. *Flora* (see color plate No 4) shows a work of art using this device. The painting depicts Flora, the goddess of life, at two different instants in time. In the surrounding flat sphere Flora stands by a window with blooming trees. In the enclosed flat sphere, Flora walks into the adjacent room where a rocking chair awaits her.

A polar image can also show the observer's movement. The image of each flat sphere can represent the visual world from a different location, revealing that the observer has changed his point of view. In this case also, although the representation of time is an integral part of the polar image, the time sequence is represented in a discontinuous fashion. We jump from one moment in time to another, visually crossing the border between one flat sphere and the next.

III. ECCENTRIC POLAR IMAGES

The previous section explained how flattening a string of continuous spherical images produces a polar image with concentric flat spheres. For the same reasons that it is possible to create a flat sphere image inside the central vanishing point of another image, it is also possible to create a flat sphere image inside any of

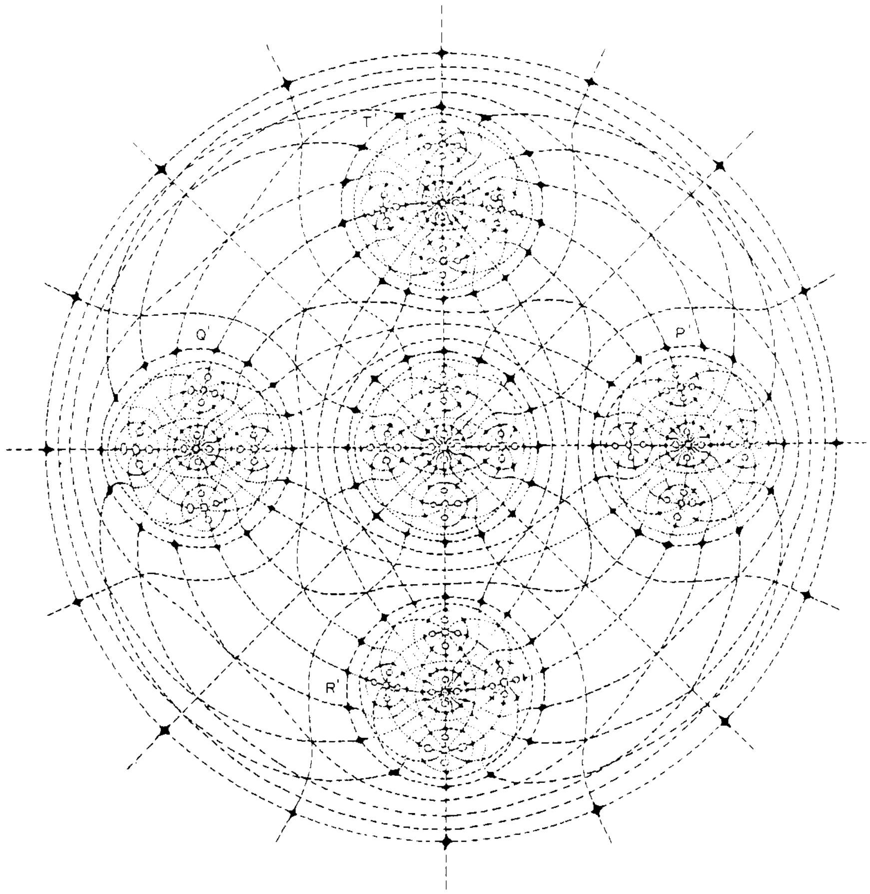

Figure 5. Perspective grid of a polar image with many concentric and eccentric flat spheres.

the vanishing points of another flat sphere image. Proceeding in this fashion, we create eccentric polar images. Figure 5 illustrates the perspective grid of a polar image with many concentric flat spheres and many eccentric flat spheres.

Now we are in a position to explain the shortcoming of flat sphere perspective that polar perspective remedies. When a single flat sphere is flattened, as in the enclosed flat sphere of Fig. 4, one of the vanishing points of the image, point N' of Fig. 4, undergoes a profound alteration. On the spherical surface, before the flattening process, point N' adequately represents a vanishing point because the grid lines converge on the point. But after the spherical image has been flattened, the lines of the perspective grid diverge on this point. Point N' is transformed from a point of convergence into a point of divergence. The graphical appearance of point N' has changed, making it different from the graphical appearance of the other five vanishing points, which remain as points of convergence.

When we build a polar perspective image, however, this disparity disappears. Notice, for example, in the polar perspective grid of Fig. 5, that in every case the divergent vanishing point of an enclosed flat sphere becomes a point of convergence relative to the surrounding flat sphere. Thus, in principle, in a polar image all vanishing points behave consistently in that they are points of convergence in one flat sphere and points of divergence in another [6]. The three-dimensional model of the grid in Fig. 5 consists of a stack of many (perhaps infinite) spherical images, each having neighboring spherical images on all six sides. A simplified model of this arrangement consists of six spherical images connected in the manner illustrated in Fig. 6.

IV. POLAR IMAGES OF MORE THAN THREE DIMENSIONS

A simple polar image of the type so far discussed represents a three-dimensional space. Implicit in those images is a fourth spatial dimension (4-D). This dimension is clearly depicted in Fig. 3 as dimension Pz. Remember that each sphere of vision contains on its surface a perspective image of a three-dimensional world. In the construction of a polar image, these spheres of vision, along with their respective images, are connected along dimension Pz. Dimension Pz is different from the three dimensions contained on the spherical images. Pz is also different from the two dimensions of the spherical

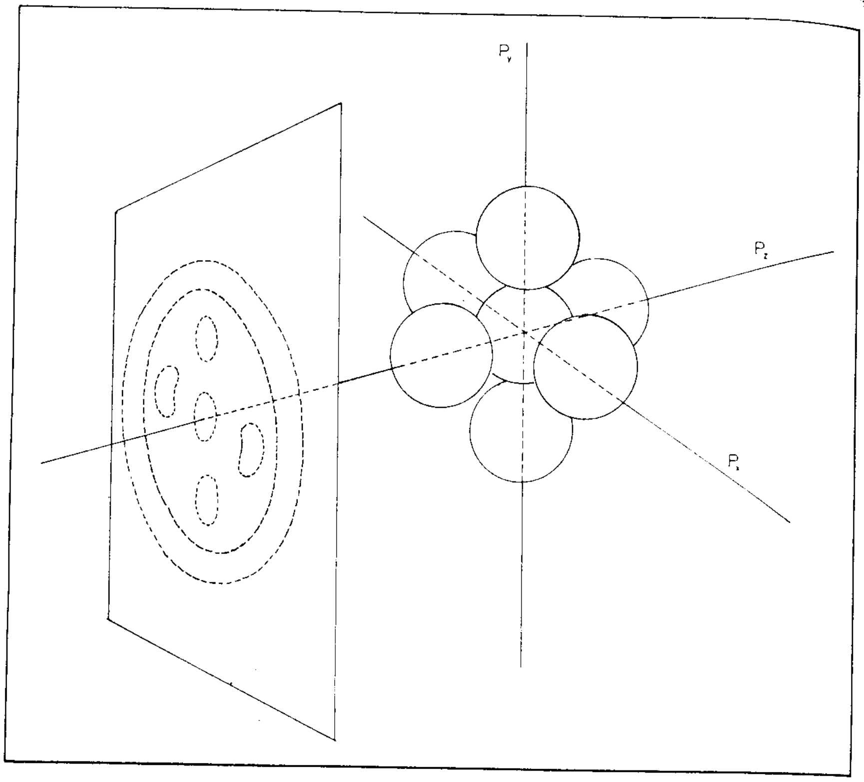

Figure 6. Seven spheres of vision connected by shared points along dimensions Pz, Py and Px.

surfaces themselves. Yet dimension Pz is a dimension that necessarily, if only implicitly, enters into the construction of a polar image. After all, a polar image is a construction that is able to connect two or more flat sphere images in a continuous and unified manner, precisely by connecting these flat spheres along a spatial dimension different from any of the spatial dimensions contained on the flat spheres.

To create a four-dimensional image, Pz must be drawn alongside the original three dimensions of the image in a coherent and nonambiguous manner. In other words, if dimension Pz can function as a fourth dimension relative to the other three, then we should be able to draw dimension Pz in a polar image without confusing it with any of the other three dimensions or upsetting the initial three-dimensional world. Figures 7 and 8 demonstrate that these two conditions are met by the mapping of dimension Pz.

First, let us recall that the depth lines of each flat sphere (the lines that go from N' to S' to N', etc.) must stretch in a circular fashion at the borders between one flat sphere and the next. When we map dimension Pz onto a polar image, the obvious danger is of confusing the lines of dimension Pz with the depth lines. Notice in Fig. 7 how these two groups of lines are not confused with each other. This is because the 4-D lines do not have to stretch in the same manner as the N'-S'-N' lines. In fact, the 4-D lines are able to run unaltered from one flat sphere to the next. This is because the Pz axis (or any line parallel to it) does not have to cover the whole of graphical points N', S', N', etc., in its path. It is this difference in appearance and behavior between the lines parallel to Pz and the depth lines of each flat sphere that makes possible the construction of an unambiguous four-dimensional image.

Figure 8 is an example of a four-dimensional image. This painting shows two flat sphere images making up a polar image of a room. This room represents our familiar three-dimensional world. But over and above this three-dimensional world, we see a man and his dog, both in the room (in the three-dimensional world) and projected along a fourth representational dimension [7]. Notice that this dimension appears unambiguously as a fourth representational dimension of the perspective system. This new fourth dimension does not alter the original three-dimensional world; rather, this fourth dimension is integrated in the image.

Figure 7 illustrates the perspective grid of four-dimensional images such as that of Fig. 8. The grid displays four sets of lines: (1) the width lines, which extend across vanishing points P' and Q' while remaining within their respective flat spheres; (2) the height lines, which extend across vanishing points T' and R' while

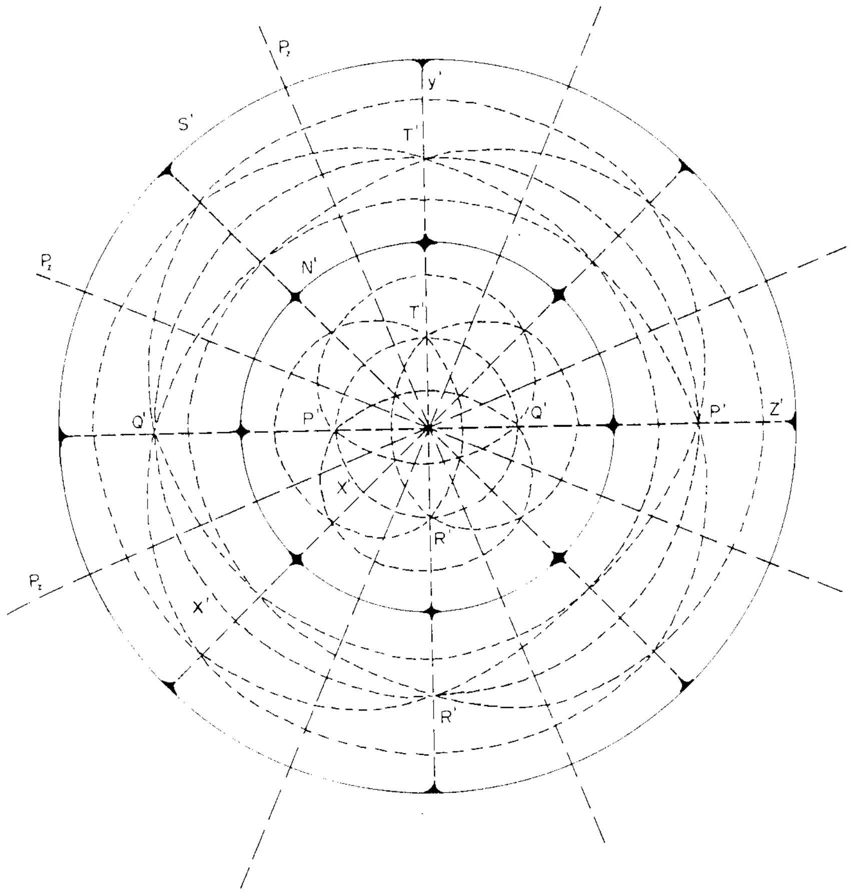

Figure 7. The perspective grid of a four-dimensional image like the one shown in Fig. 8. This grid exhibits four distinct sets of lines. The fourth-dimensional lines are those that cross the boundaries between one flat sphere and another unaltered.

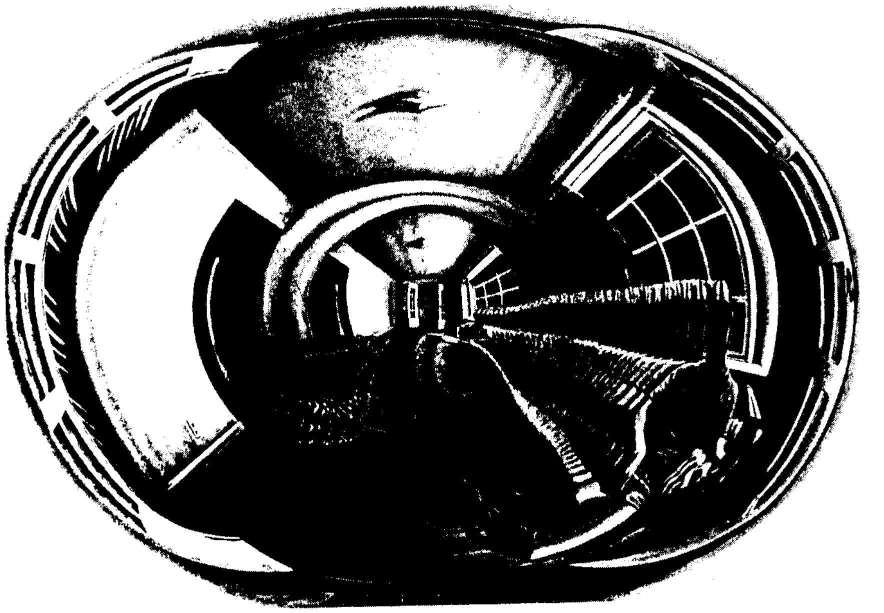

Figure 8. *Stephen and Rufus*, oil on panel, 70 x 48 inches, 1982. A man and his dog project along a fourth dimension representing their movement through time. To construct the figure, one of the two flat spheres is turned inside out, a necessary inversion for the flat spheres to connect in the manner represented.

remaining within their respective flat spheres; (3) the depth lines, which extend across vanishing points N' and S' while remaining within their respective flat spheres, and (4) the four-dimensional lines, which extend across the whole of the polar image and do not have to remain contained within the individual flat spheres. They start at vanishing point S' at the center of the image and extend outward. (These lines will eventually vanish at another point N' not represented in this grid.)

We have seen that dimension Pz works graphically as a fourth dimension relative to the other three. It can be mapped in a consistent and unambiguous manner together with the three original dimensions of the flat spheres. But what reason do we have to consider dimension Pz as a dimension perpendicular to the three dimensions contained within each flat sphere?

Dimension Pz is perpendicular to the spherical surfaces that represent the sphere of vision. These spherical surfaces create the illusion—for an observer—of a three-dimensional world. Thus, each spherical surface contains a three-dimensional, purely illusionary world. We take, then, a dimension Pz, which is perpendicular to these spherical surfaces, as a dimension perpendicular to the three dimensions contained in each of those surfaces [8].

Figure 6 is the model of a polar image with concentric and eccentric flat spheres representing seven spheres of vision. Imagine that we map the six outer spheres of vision onto the surface of the central sphere of vision. We obtain one sphere of vision that contains six 'flat spheres' on its surface. This mapping introduces into the image of the central sphere the dimensions Px, Py and Pz represented in Fig. 6. These three lines are actually four-dimensional lines relative to the three dimensions contained in the central sphere of vision. This is so because these three lines are perpendicular to the surface of the central sphere of vision, as is any other line that is the radius of the central sphere.

A hypercube (Fig. 9) further illustrates the construction of polar four-dimensional images. In it we can see a body made of eight cubes: one cube appearing in each of the two flat spheres, and six more cubes created by the faces of the first two cubes when these faces are consistently connected along a fourth dimension. In spite of its first appearance, this image is not to be read as a cube inside a cube. The cube in the enclosed flat sphere is not inside the cube of the surrounding flat sphere. Rather, the

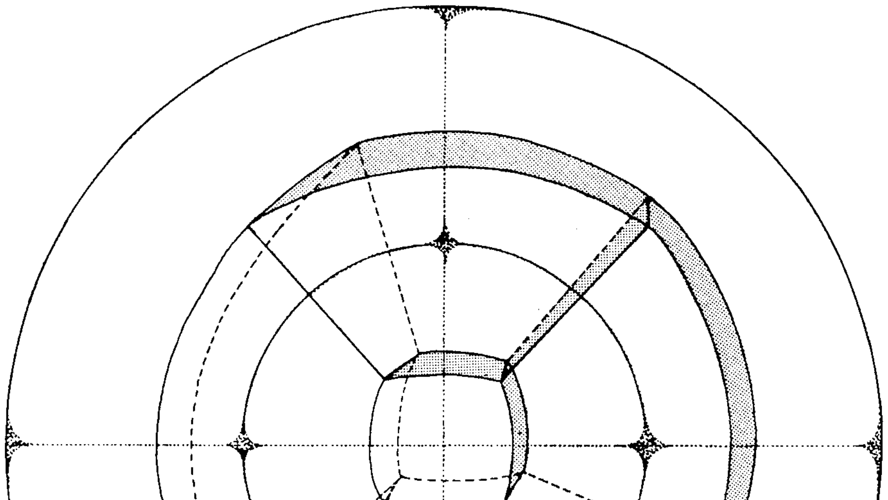

Figure 9. A hypercube drawn with polar perspective. Two flat spheres, each containing a diagram of a cube, form a polar image. The vertices of each of the two cubes are connected along a fourth dimension, creating a hypercube. An inversion, similar to that mentioned in Fig. 8, was introduced to create this figure.

perspective lines indicate that the first cube is further away from the observer than the second cube along a fourth dimension, as explained above.

Acknowledgements — Some of the ideas presented in this paper developed from long discussions with my friend Bruce Leutwyler. In fact, it was he who first saw the potential of polar perspective for representing more than three spatial dimensions. His critical and substantive insights have been a constant guide to my work on perspective, and he is largely responsible for the existence of polar perspective. I am also grateful to Professor J. S. Fulton for his constructive observations and to Steve Adams for his constant support and help with the manuscript.

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1. For a thoughtful discussion of the column paradox, see E. H. Gombrich, *Art and Visual Illusion* (Princeton: Princeton University Press, 1972) pp. 254–256.
2. R. Veru, *Understanding Perspective* (New York: Van Nostrand Reinhold, 1980).
3. F. R. Casas, "Flat Sphere Perspective", *Leonardo* **16**, 1-9 (1983).
4. Artist Richard Termes of South Dakota has also worked out the six-point spherical perspective structure. Mr Termes uses the system to paint images on the exterior surface of large spheres. K. R. Adams has described another method for depicting the entire sphere of vision ("Tetrasonic Perspective for a Complete Sphere of Vision", *Leonardo* 9, 289-291, 1976). However, his tetraconic system creates a discontinuous image, an image broken up in discrete facets.
5. M. C. Escher produced several images that indicate this limitation of visual field. Some of his prints show the convex image he saw reflected on one side of a mirror ball. He did not create any image that could depict the entire visual space around or reflected by the mirror ball.

   In the field of perspective, Escher's most substantial contribution, to my knowledge, is the development of cylindrical perspective. Cylindrical perspective results from projecting the three spatial dimensions onto a portion of a cyclindrical surface. This projection creates a grid with three vanishing points of convergence, as is clearly evident in his print *House of Stairs* and even more evident in the perspective grid he prepared for the construction of that print. This grid is reproduced in J. L. Locher, ed., *The World of M. C. Escher* (New York: Harry N. Abrams, Inc., 1971).
6. Any *actual* polar image we construct, however, will be inconsistent in this regard because the polar image will have a finite number of flat spheres. Consequently, the flat sphere at one end of the finite sequence will have only a point of divergence and the flat sphere at the other end of the sequence, only a point of convergence.
7. The four-dimensional lines in Fig. 8 have been cut in many three-dimensional slices. This was done for aesthetic and historical reasons. The four-dimensional lines in polar perspective can be drawn with solid, unbroken lines.
8. It is possible to represent a fourth dimension only when the three dimensions on the image surface have an equal status and are independent of the dimensions of the image surface itself. If one or two of the dimensions of the image coincide with either of the two dimensions of the image surface, as is the case in classical perspective and cylindrical perspective, then a dimension perpendicular to the image surface would be ambiguous. It could not then represent a fourth dimension relative to the three dimensions of the image. In classical perspective, two dimensions of the image are not independent of the dimensions of the surface on which they appear. Therefore, a dimension perpendicular to the plane of representation is actually the third—the depth—dimension of the image. In cylindrical perspective, only one of the dimensions of the image is not independent of the dimensions of the representational plane. The other two dimensions are wholly illusionary. Therefore, a dimension perpendicular to the cylindrical surface (or to the flat surface after the cylindrical image has been flattened) is a dimension different from the other three dimensions of the image, but not equally different. This dimension is actually perpendicular only to the dimension of the image that coincides with one of the dimensions of the image surface, but it is perpendicular to the other two dimensions of the image in a purely illusionary sense.

   It may be important at this point to clarify the status of the fourth graphical dimension in relation to the other three. The first three dimensions are equally illusionary dimensions relative to the image surface on which they appear. But polar dimension Pz is not an illusionary dimension relative to this surface. It is actually a dimension perpendicular to the image surface. From a purely graphical point of view, this disparity between Pz and the other three dimensions is no more troubling than the disparity in classical perspective between the depth dimension which is purely illusionistic and the other two dimensions of the image which are not independent of the representational surface.